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LETTER TO THE EDITOR

Structural instability and the massless $\lambda\varphi^4$ theory

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Abstract. It is shown that the massless $\lambda\varphi^4$ theory is structurally unstable, and quantum fluctuations cause symmetry breaking and the unfolding of the cusp catastrophe.

It has recently been shown that quantum fluctuations can act as the driving mechanism for spontaneous symmetry breaking in the massless $\lambda\varphi^4$ theory (Ghose 1982, 1983). The purpose of this note is to comment on the topological significance of the result already established by detailed calculations (up to two loops). I shall argue that the massless $\lambda\varphi^4$ theory is 'structurally unstable' with a degenerate critical point at the origin which is removed by quantum fluctuations. The resultant effective potential is a universal unfolding of the critical point of $\lambda\varphi^4$; it is the potential of the cusp catastrophe.

We shall first define a few terms and state some fundamental and powerful results of local differential topology. The interested reader is referred to Poston and Stewart (1978) for details. A function $f: R^n \to R$ is said to have a *critical point* at u if $Df|_u = 0$ where D denotes the derivative (defined as the best approximating linear map at u). The critical point u is said to be *non-degenerate* if D^2f is non-singular. Functions which have only non-degenerate critical points are called *Morse functions*. Two functions f and g are said to be *equivalent* around the origin 0 if there exists a local *diffeomorphism* (a smooth reversible change of coordinates) $y: R^n \to R^n$ around 0 and a constant γ such that $g(x) = f(y(x)) + \gamma$ around 0; the shear term γ adjusts the value of the function at 0. A function f is said to be *structurally stable* if under sufficiently small and smooth perturbations p, the functions f and $f+p$ are *equivalent*; in other words, if the critical points of f and $(f+p)$ are of the same type. Now, there is a powerful result which says *only Morse functions are structurally stable* (Poston and Stewart 1978). For example, the function x^2 is structurally stable because no perturbation, if it is sufficiently small, would change its behaviour. For example $x^2+2\varepsilon x$ (ε an arbitrary small constant) can be written as $u^2-\varepsilon^2$ with $u = x+\varepsilon$; only the Morse critical point is shifted from 0 to $-\varepsilon$. On the other hand, the functions x^3 and x^4 are non-Morse and unstable because there are polynomials arbitrarily close to them which are not of the same type. For example, the function $(x^4+\varepsilon x^2)$ has one maximum at $x = 0$ and two minima at $x = \pm(-\frac{1}{2}\varepsilon)^{1/2}$ for arbitrarily small but negative ε ($\varepsilon < 0$), whereas the function x^4 has a single minimum at $x = 0$.

In addition to structurally stable single functions one needs also to consider smooth *families* of functions (which may include individual functions with degenerate critical points) parametrised by *control parameters* $(a, b, \ldots)$. Two r-parameter families of functions $f, g: R^n \times R^r \to R$ are *equivalent* if there exist in the neighbourhood of the origin (i) a diffeomorphism $e: R^r \to R^r$, (ii) a smooth map $y: R^n \times R^r \to R^n$ such that for each $s \in R^r$ the map $y_s(x) = y(x, s)$ is a diffeomorphism, and (iii) a smooth map $\gamma: R^r \to R$ such that $g(x, s) = f(y_s(x), e(s)) + \gamma(s)$ for all $(x, s) \in R^n \times R^r$ in that neighbourhood. If $f: R^n \times R^r \to R$ is equivalent to any family $f+p: R^n \times R^r \to R$ for a sufficiently small family $p: R^n \times R^r \to R$ of perturbations, then f is structurally stable. An example of a structurally stable family is the cusp catastrophe $V(x) = x^4 + ax^2 + bx$. (The cubic and constant terms can always be eliminated by a suitable choice of the origin.) $V(x)$ is called the *universal unfolding* of the critical point of x^4 which has three coincidental critical points that separate under a small perturbation. This is a very powerful result. It is universal in the following sense: any smooth function of x' and parameters $(a', b'; c, d, \ldots)$ which has the form $kx'^4 +$(higher-order terms in x') (with $k \rightharpoonup 0$) when $a' = b' = c = d = \ldots = 0$ can always be converted into the form $V(x)$ by a suitable smooth coordinate substitution $x = x(x'; a', b', c, d, \ldots)$, $a = a(a', b', c, d, \ldots)$, $b = b(a', b', c, d, \ldots)$ valid in some neighbourhood of $(0; 0, 0 \ldots)$. In other words, $V(x)$ captures completely the effects of *all* possible unfoldings and *all* possible perturbations; it gives all the types near anything equivalent to x^4 and the geometry of how they develop under a continuously varying perturbation.

Our examples have been polynomials, but this is no real restriction. Any sufficiently smooth function of one variable can be expanded as a formal Taylor series. If its critical point is degenerate (i.e. the second derivative vanishes) and its first non-vanishing derivative is the nth-order derivative (with n finite), then the critical point has $(n-2)$-fold degeneracy and it requires $(n-2)$ unfolding (or control) parameters. The critical point is said to be of *codimension* $(n-2)$. For example, the critical point of x^4 is of codimension 2 and requires two unfolding parameters. The behaviour of the rest of the terms of the series or whether the series converges does not matter because we are concerned with what happens arbitrarily close to the origin. Consider, for example, the function $x^4 + \alpha x^5$; the additional critical point occurs at $x = -(4/5\alpha)$ which is arbitrarily far from the origin for arbitrarily small α. In Zeeman's well known words, the tail (of the Taylor series) cannot wag the dog! There is a powerful result here: an algorithm exists for deciding whether or not the Taylor series of a function $f: R^n \to R$ up to order k (called the k-jet $j^k f$) is sufficient to determine the behaviour of the function close to the origin. For a single variable the result is elementary. (For details, see Poston and Stewart (1978).)

In this connection a recent paper by Miller (1982) is worth drawing attention to. He argues that no perturbative calculation can discover the true minima of a potential. It is well known that there are smooth functions which are not well approximated by their formal expansions. For example, $f(x) = \exp(-1/x^2)$ for $x \neq 0$, $f(x) = 0$ for $x = 0$ is of infinite codimension and has the well defined expansion $f(x) \doteq f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \cdots = 0 + 0 + 0 + \cdots$ which coincides with the Taylor expansion of $f(x) \equiv 0$; however, the two functions have only one common point $(x = 0)$. To look for isomorphisms between functions and Taylor expansions is therefore a red herring. Fortunately, in order to study the effect of small perturbations (e.g. quantum fluctuations) on a function near the origin one need only look at the critical points of the function and see if they are structurally stable or unstable but capable of unfolding into structurally stable families with a finite number of unfolding parameters (i.e. of finite codimension).

The above considerations tell us that the massless $\lambda\varphi^4$ theory is structurally unstable, and I shall show in this paper that the perturbatively *reliable* asymmetric solution found earlier by explicit computation (up to two loops) has structural stability in the presence of an external c-number source. No further (small) perturbations (e.g. higher loops) can alter this qualitative topological character: the universal unfolding of the cusp catastrophe occurs already in the loop approximation. The worst thing that can happen topologically to φ_c^4 near the origin $\varphi_c = 0$ already happens in the loop approximation: φ_c^6 and higher-order terms can be rigorously ignored near $\varphi_c = 0$.

It is also clear now why Coleman and Weinberg (1973) missed this stable asymmetric solution. Their renormalisation condition $\partial^2 V/\partial\varphi_c^2|_0 = 0$ forces the critical point of φ_c^4 to remain degenerate: the unfolding is prevented from occurring. No wonder then that the minimum that appeared in their one-loop calculation is false! The lesson is that not all renormalisation conditions are topologically equivalent and one must bear this in mind in choosing them.

The effective potential (see Coleman and Weinberg (1973)) of the massless $\lambda\varphi^4$ theory coupled to an external c-number source J is given by

$$V(\varphi_c, J) = (\lambda + c)\varphi_c^4/4! + \tfrac{1}{2}B\varphi_c^2 - J\varphi_c, \tag{1}$$

where B and C are counterterms which can be determined by imposing the renormalisation conditions (Ghose 1983)

$$\mathrm{d}^4 V/\mathrm{d}\varphi_c^4|_M = \lambda, \tag{2}$$

$$\mathrm{d}^2 V/\mathrm{d}\varphi_c^2|_0 = f(\hbar)b, \tag{3}$$

where M is an arbitrary renormalisation mass, $f(\hbar)$ is any function of the loop expansion parameter $\hbar$ that satisfies $f(0) = 0$, $f(1) = 1$, guaranteeing a massless theory in the tree approximation, and b is to be determined by requiring the broken symmetric solution to be perturbatively reliable. Notice that there is no symmetry in the theory which requires b to vanish except scale invariance which is afflicted by anomalies. One finds in the one-loop approximation†

$$V = \frac{\lambda}{4!}\varphi_c^4 + \frac{\hbar\lambda^2\varphi_c^4}{256\pi^2}\left(\ln\frac{\varphi_c^2}{M^2} - \frac{25}{6}\right) + \tfrac{1}{2}f(\hbar)b\varphi_c^2 - J\varphi_c. \tag{4}$$

The broken symmetric solution $\langle\varphi\rangle$ obtained by requiring

$$\left.\frac{\partial V}{\partial\varphi_c}\right|_{\substack{\langle\varphi\rangle\neq 0\\ J=0}} = 0 \tag{5}$$

is given by

$$\lambda\ln(\langle\varphi\rangle^2/M^2) = -(64\pi^2/\hbar\lambda)(\tfrac{1}{6}\lambda + f(\hbar)b/\langle\varphi\rangle^2) + \tfrac{11}{3}\lambda. \tag{6}$$

It is clear that $\langle\varphi\rangle$ vanishes in the tree approximation ($\hbar = 0$), guaranteeing a symmetric massless theory in that approximation, and the asymmetric solution is reliable ($|\lambda| \ll 1$, $|\lambda \ln \varphi_c/M| \ll 1$) provided (remembering $f(1) = 1$)

$$b = -\tfrac{1}{6}\lambda\langle\varphi\rangle^2 + \mathrm{O}(\lambda^2). \tag{7}$$

The $\mathrm{O}(\lambda^2)$ term can be fixed by requiring $f(\hbar)b$ to be renormalisation group invariant

† The last-but-one term would be absent if one were to use the renormalisation condition $\partial^2 V/\partial\varphi_c^2|_0 = 0$ used by Coleman and Weinberg (1973), which, as remarked earlier, would force the critical point at $\varphi_c = 0$ to remain degenerate and prevent the unfolding from occurring.

(Ghose 1983). Equations (3) and (7) show that the critical point is now non-degenerate: the minimum of the classical potential at $\varphi_c = 0$ has been turned into a maximum and there are two new minima at $\pm\langle\varphi\rangle$.

Now consider the renormalisation group equation

$$(M\partial/\partial M + \beta\partial/\partial\lambda + \gamma\varphi_c\partial/\partial\varphi_c)V = 0. \quad (8)$$

Following Coleman and Weinberg (1973), consider the dimensionless quantity $V^{(4)} = \partial^4 V/\partial\varphi_c^4$ and define the dimensionless variables

$$t = \ln \varphi_c/M, \qquad \bar{\beta} = \beta/(1-\gamma), \qquad \bar{\gamma} = \gamma/(1-\gamma). \quad (9)$$

Then $V^{(4)}$ satisfies the equation

$$(-\partial/\partial t + \bar{\beta}\partial/\partial\lambda + 4\bar{\gamma})V^{(4)}(t, \lambda) = 0. \quad (10)$$

In the one-loop approximation,

$$\beta = 3\hbar\lambda^2/16\pi^2, \qquad \gamma = 0, \quad (11)$$

and so (10) becomes

$$(-\partial/\partial t + \beta\partial/\partial\lambda)V^{(4)}(t, \lambda) = 0. \quad (12)$$

Define $\lambda''(t, \lambda)$ by

$$\mathrm{d}\lambda''/\mathrm{d}t = \bar{\beta}(\lambda'') = 3\hbar\lambda''^2/16\pi^2 \quad (13)$$

with $\lambda''(0, \lambda) = \lambda$. Then

$$\lambda'' = \frac{\lambda}{1 - 3\hbar\lambda t/16\pi^2} \underset{t\to-\infty}{\sim} -\frac{16\pi^2}{3\hbar t}. \quad (14)$$

Thus λ drops out near the origin of classical field space and is replaced by $\langle\varphi\rangle$. This is called '*dimensional transmutation*'.

Now define the auxiliary quantity

$$\lambda' = \frac{\lambda}{1 - (3\hbar\lambda/32\pi^2)(\ln \varphi_c^2/M^2 - \frac{25}{6})}. \quad (15)$$

In terms of λ' the renormalisation group 'improved' effective potential can be written down as

$$V = (\lambda'/4!)\varphi_c^4 + \tfrac{1}{2}f(\hbar)b\varphi_c^2 - J\varphi_c \quad (16)$$

$$= \frac{\lambda}{4!}\varphi_c^4 + \frac{\hbar\lambda^2\varphi_c^4}{256\pi^2}\left(\ln\frac{\varphi_c^2}{M^2} - \frac{25}{6}\right) + \tfrac{1}{2}f(\hbar)b\varphi_c^2 - J\varphi_c + \ldots. \quad (17)$$

It is clear from (17) that the one-loop approximation is reproduced in its expected domain of validity, namely $|\lambda| \ll 1$, $|\lambda \ln \varphi_c/M| \ll 1$; equation (16) is, however, valid over the much wider range $-\infty < t < \infty$ but only to lowest order in $\hbar$.

Noticing that

$$\partial\lambda'/\partial\varphi_c = (3\hbar/16\pi^2)(\lambda'^2/\varphi_c), \quad (18)$$

we get

$$\left.\frac{\partial V}{\partial\varphi_c}\right|_{\substack{\langle\varphi\rangle\neq 0\\ J=0}} = \left(\frac{f(\hbar)b}{\langle\varphi\rangle^2} + \tfrac{1}{6}\langle\lambda'\rangle + \mathrm{O}(\lambda'^2)\right)\langle\varphi\rangle^3 = 0. \quad (19)$$

Thus to lowest order in λ',

$$f(\hbar)b = -\tfrac{1}{6}\langle\lambda'\rangle\langle\varphi\rangle^2. \tag{20}$$

Using this result in (16), we get

$$V = (\lambda'/4!)\varphi_c^4 - \tfrac{1}{12}\langle\lambda'\rangle\langle\varphi\rangle^2\varphi_c^2 - J\varphi_c. \tag{21}$$

Notice that

$$V(\langle\varphi\rangle, J=0) = -(\langle\lambda'\rangle/4!)\langle\varphi\rangle^4 < 0. \tag{22}$$

Since $V(0,0)=0$, $\langle\varphi\rangle$ is an absolute minimum.

Now define a new field variable

$$\varphi_c' = (\lambda'/3!)^{1/4}\varphi_c. \tag{23}$$

The effective potential (21) can be rewritten in terms of φ_c' as

$$V(\varphi_c', J') = \tfrac{1}{4}\varphi_c'^4 - \tfrac{1}{2}a\varphi_c'^2 - J'\varphi_c' \tag{24}$$

with

$$a = (\langle\lambda'\rangle/\lambda')^{1/2}\langle\varphi'\rangle^2, \qquad J' = (3!/\lambda')^{1/4}J. \tag{25}$$

In differentiating V with respect to φ_c' both J' and a can be treated as constants in the neighbourhood of the origin ($t \to -\infty$). Therefore (24), being in the standard form of a 'cusp catastrophic' potential, is a structurally stable universal unfolding with J' and a as unfolding parameters or control variables (J' being the 'normal factor' and a the 'splitting factor') and φ_c' the state variable. The bifurcation set is given by

$$27J'^2 = 4a^3. \tag{26}$$

Equation (24) has been arrived at via various approximations. Nevertheless, it has *universality* in the following sense: any smooth function of φ_c'' and parameters $J'', a'; b, c, \ldots$ which has the form ($k \to 0$) $k\varphi_c''^4$ + (higher-order terms in φ_c'') when $J'' = a' = b = c = \cdots = 0$ can be converted into the form (24) by a coordinate substitution

$$\varphi_c' = \varphi_c'(\varphi_c''; J'', a', b, c, \ldots), \qquad J' = J'(J'', a', b, c, \ldots), \qquad a = a(J'', a', b, c, \ldots), \tag{27}$$

valid in some neighbourhood of $(0; 0, 0, \ldots)$. Therefore, given that the *critical point structure* of (24) is of physical interest, catastrophe theory tells us that we would have the *same* critical point structure (after a coordinate change (26)) *even if* $\varphi_c'^6$ *and higher terms were included*, for small values of φ_c' and the parameters a and J'.

The usual massive $\lambda\varphi^4$ theory has an additional parameter λ, and therefore cannot be thrown into the standard cusp catastrophic form (24). It is 'dimensional transmutation' (replacement of λ by $\langle\varphi\rangle$) which allows the cusp catastrophe to occur in the case of the massless $\lambda\varphi^4$ theory.

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